Rubber bushing · Dynamic Stiffness · Carbon black · Amplitude dependence · Fletcher-Gent effect · Bushing measurement

A simple engineering model for the axial dynamic stiffness of carbon black filled rubber bushings in the frequency domain including amplitude dependence is presented. The axial stiffness is calculated by applying a newly developed rubber model to an equivalent strain of the strain state inside the bushing, thus yielding an equivalent shear modulus which is inserted into an analytical formula for the stiffness. Therefore, this procedure includes the Fletcher-Gent effect inside the bushing due to non-homogeneous strain states. An experimental verification is carried out on a commercially available bushing. Firstly, the model parameters are obtained from two different experiments: elastic and friction parameters are achieved from one quasi-static test performed at 0.1 Hz and 3 mm amplitude, while frequency dependence parameters are obtained from one dynamic test at 0.1 mm of amplitude over a frequency range from 1 to 100 Hz. Secondly, new dynamic tests at five amplitudes are performed on the same bushing and the results are compared to those of the model showing good agreements.

Axial Stiffness of Carbon Black Filled Rubber Bushings

Frequency and Amplitude Dependence

Since the comfort inside the vehicles offer the manufactures the possibility for achieving differentiation from other companies, to decrease the unwanted vibrations that go from the road to the inside of the vehicles has become a subject of special interest. One way to manage it is the use of rubber bushings within the suspension systems. The main characteristic of rubber materials is their elasticity and capacity for dissipating vibration energy which is represented by the hysteresis loop enclosed by the loading and unloading curves in a stress-strain diagram. Furthermore, rubber materials are strongly dependent on amplitude, known as the Fletcher-Gent [1] effect, which becomes more pronounced with the presence of rubber fillers such as carbon black, producing a distortion of the hysteresis loop as reviewed by Medalia [2]. Several models have been developed to represent this property, like Kraus [3] who explains the amplitude dependence as due to the continuous breaking and reforming of van der Waals bonds between carbon-black aggregates, while modifications and experimental application are carried out by Ulmer [4] and Lion [5] presenting a time domain formulation of the Kraus model. In addition, the Fletcher-Gent effect is also characterized using friction models, like Gregory [6], Coveney et al. [7] and Kaliske and Rothert [8] proposing the Prandtl element, a Coulomb damper in series with an elastic spring; a model expanded by Bruni and Collina [9], Austrell et al. [10] and Brackbill et al. [11]. Furthermore, several authors have employed these material models in order to calculate the dynamic stiffness of rubber bushings. Some of them insert the models into finite element systems, like Austrell et al. [12] who represent frequency and amplitude dependencies adding integer derivatives to stick-slip friction components, similar to Gil-Negrete [13] except for the use of fractional derivatives. However, finite element procedures require a long-time overlay process to calculate the stiffness. In contrast, other authors work directly at the component level, like Berg [14] who presents a five-parameters model which gives a good resemblance to the smoother-friction rubber behavior, also used by Sjöberg and Kari [15] together with a rate-dependent part using fractional derivatives, or Misaji et al. [16] who take into account amplitude dependence as the parameters of an ordinary Kelvin-Voigt model are updated continuously for every oscillation cycle. However, the latter methods neglect the amplitude dependence inside the rubber bushing due to non-homogeneous strain states. This paper presents a simple engineering formula to predict the axial dynamic behavior of rubber bushings by extending to the axial case the newly developed torsion model by García Tarrago M.J. et al. [17]. The frequency domain dynamic stiffness including the amplitude dependence is obtained by applying a rubber model to an equivalent strain of the strain state inside the bushing thus yielding an equivalent modulus which is inserted into an analytical formula for the stiffness. The model result is the one of applying a separable elastic

Axiale Steifigkeit von mit Ruß gefüllten Gummibüscheln einschließlich Frequenz- und Amplitudeabhängigkeit

Gummibüschel · Dynamische Steifigkeit · Ruß · Amplitudendependenz · Fletcher-Gent-Effekt · Bushenmessungen

Ein einfaches Ingenieurmodell für die axial-dynamische Steifigkeit von mit Ruß gefüllten Gummibüscheln wird gezeigt. Das Modell arbeitet in einem Frequenzbereich der die Amplitudeabhängigkeit einschließt. Die axiale Steifigkeit wird durch die Anwendung des Gummimodells auf eine äquivalente Dehnung des Dehnungszustandes innerhalb der Buchse berechnet, wodurch sich ein äquivalentes Schermodul ergibt, dass in der analytischen Formel für die Steifigkeit eingesetzt wird. Daher wird in dieser Methode der Fletcher-Gent-Effekt innerhalb der Buchse aufgrund eines nicht-homogenen Dehnungszustandes einge- baut. Eine experimentelle Verifizierung wird an einer kommerziellen Buchse durchgeführt. Zunächst werden die Modellparameter aus zwei verschiedenen Experimenten gewonnen: Elastizitäts- und Reibungsparameter werden aus einem quasistatischen Test mit einer Frequenz von 0,1 Hz und einer Amplitude von 1 mm durchgeführten Tests erlangt, während die frequenzabhängigen Parameter aus einem dynamischen Test mit 0,1 mm Amplitude über einen Frequenzbereich von 1 bis 100 Hz gewonnen werden. Drauf und zurück werden neue dynamische Tests mit fünf verschiedenen Amplituden an der gleichen Buchse durchgeführt und die Ergebnisse mit denen des Modells verglichen, wobei eine gute Übereinstimmung erreicht wird.

Autoren

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tic, viscoelastic and friction component model to the material level while using only five parameters. An experimental verification is carried out on a commercial bushing. Firstly, the model parameters are obtained from two different tests: elastic and frequency independent friction parameters are extracted from the hysteresis loop of a large deformation quasi-static test while the viscoelastic parameters are achieved from one dynamic test performed over a frequency range from 1 to 100 Hz. Secondly, new dynamic tests at five amplitudes are carried out on the same bushing and the results are displayed together with the model stiffness showing a good agreement thus verifying the accuracy of the presented method including the amplitude and frequency dependence.

Model

Rubber model

The recently developed rubber material model [17] applied in this paper consists of three parallel components while using only five parameters, see Figure 1. The first branch represents the elasticity by a linear relation between the elastic stress $\sigma_{\text{elas}}$ and the strain $\varepsilon$ where $\mu$ is the elastic shear modulus and $t$ the time,

$$\sigma_{\text{elas}}(t) = 2\mu\varepsilon(t)$$  \hspace{1cm} (1)

The second component in Figure 1 indicates the frequency dependence and it is represented by fractional derivatives which increase the ability to adjust to measured characteristics while increasing the number of parameters to only two: a proportional constant $m$ and the time derivative order $\alpha$,

$$\sigma_{\text{elas}}(t) = mD^\alpha\varepsilon(t) \quad 0 < \alpha \leq 1$$  \hspace{1cm} (2)

where $\sigma_{\text{elas}}$ is the viscoelastic stress, $D^\alpha$ denotes the fractional time derivative of order $\alpha$, defined through an analytical continuation of a fractional Riemann-Liouville integration [18]. Numerically, the viscoelastic stress component can be evaluated as,

$$\sigma_{\text{elas}}(t) \approx m\frac{\Delta t^{-\alpha}}{\Gamma(1-\alpha)} \sum_{j=0}^{n-1} \frac{\Gamma(j+1)}{\Gamma(j-\alpha)} \varepsilon_{n-j}$$  \hspace{1cm} (3)

where $\varepsilon_{n-j} = \varepsilon((n-j)\Delta t)$, $\varepsilon_0$ is the final strain at time $t_0 = n\Delta t$, $\Delta t$ is a constant time step applied in the estimation process and $\Gamma$ denotes the Gamma function [19] defined as $\Gamma(\beta) = \int_1^\infty x^{\beta-1}e^{-x}ds + \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\beta^k}$ for $\beta \neq 0$, $-1$, $-2$, .... The third branch represents the amplitude dependence by a smooth friction component which enables a very good fit to measured curves using only two parameters $\sigma_{\text{max}}$ and $\varepsilon_{1/2}$. The frictional stress $\sigma_{\text{frict}}$ develops gradually following the equation

$$\sigma_{\text{frict}}(t) = \sigma_{\text{frict}}(0) + \frac{1}{\varepsilon_{1/2}} \left[ 1 - \text{sign}(t) \right] \left[ \frac{\varepsilon_{\text{frict}}}{\varepsilon_{\text{frict}(0)}} + \frac{\varepsilon_{\text{frict}(0)}}{\varepsilon_{\text{frict}}(t)} - \varepsilon_{1/2} \right]$$  \hspace{1cm} (4)

where the material parameters $\sigma_{\text{frict}}$, and $\varepsilon_{1/2}$ are the maximum friction stress developed and the strain needed to develop half the maximum friction stress, with $\text{sign}(t)$ denoting the sign of the strain state. The values of $\sigma_{\text{frict}}$ and $\varepsilon_{1/2}$ are updated each time the strain changes direction at $\varepsilon = 0$ as $\sigma_{\text{frict}} := \sigma_{\text{frict}(0)}$ and $\varepsilon_{1/2} := \varepsilon_{1/2}(0)$. Finally, the total stress is the sum of these three stresses. However, a shear modulus cannot be directly achieved from the nonlinear relation between stress and strain obtained when the rubber model is applied to any strain inside the rubber, as would be the case with the elastic component. Moreover, that relation is valid at that specific strain, but in order to use it in the formula of the dynamic stiffness a “global” value for the shear modulus is required. The simplified engineering solution proposed in this paper calculates an equivalent shear strain for the whole bushing using the classical theory of elasticity to subsequently apply it to the newly material model, thus yielding an equivalent shear modulus which is inserted into an analytical formula for the axial stiffness.

Equivalent strain

The strain state inside the rubber bushing of length $L$ in Figure 2 bonded to two cylindrical layers at inner and outer radii $a$ and $b$ when it is subjected to an axial harmonic displacement at the outer surface relative to the inner one $d(t) = d\sin(\omega_0 t)$ with $\omega_0$ the excitation frequency and $d$ the amplitude, is obtained by the classical linear theory of elasticity while considering that the constitutive equation between stress and strain is made up of two components of
the rubber model: the elastic and the fractional derivative. The amplitude dependence is taken into account later on when the rubber model is applied to the equivalent strain. As a result, the strain state reads
\[ \varepsilon(r, t) = \frac{d(t)}{2 \log_e \left( \frac{b}{a} \right)}. \] (5)
where \( r \) is the radius. Subsequently, an equivalent strain is calculated following a general methodology which can be used with more complex strain states and consists in an energy density balance between the axial deformation of a rubber bushing and a simple shear specimen made of the same material giving
\[ \varepsilon_{\text{equiv}}(t) = \frac{d(t)}{\sqrt{2(b^2 - a^2) \log_e \left( \frac{b}{a} \right)}}. \] (6)

**Formula for the dynamic stiffness in the frequency domain**

Now it is time to apply the rubber model presented in Eq (1), (2) and (4), to the equivalent strain, thus leading to an equivalent total stress. In addition, the excitation frequency of the displacement signal is varied as \( \omega_b = s \Delta \omega \) with \( \Delta \omega \) a constant frequency step and \( s \in \mathbb{N} \), and therefore, \( s \) different equivalent strains and their corresponding stress are obtained. Consequently, the equivalent shear modulus at frequency \( \omega_c \) is the result of dividing the value of the temporal Fourier transform of the total stress at the excitation frequency by the temporal Fourier transform of the equivalent strain at the same frequency
\[ \mu_{\text{equiv}}(\omega_c) = \frac{\sigma_{\text{total}}(\omega_c)}{2 \varepsilon_{\text{equiv}}(\omega_c)}. \] (7)
where \( \omega_c \) denotes the temporal Fourier transform. This linearization process takes into account the non-linear relation between stress and strain at frequency \( \omega_b \) considering the first order response while omitting the less important overtones (stress response at \( 3 \omega_b, 5 \omega_b \)). Finally, the dynamic stiffness \( K \) is achieved by inserting the modulus of Eq (7) into an analytical formula for the axial stiffness calculated previously by the classical linear theory of elasticity while using the same boundary conditions and constitutive equation as to obtain the strain state of Eq (5)
\[ K(\omega_c) = \frac{2 \pi L}{\log_e \left( \frac{b}{a} \right)} \mu_{\text{equiv}}(\omega_c). \] (8)

This formula represents the dynamic behavior in the axial direction of the rubber bushing in Figure 2 including amplitude and frequency dependence.

**Experimental**

An experimental verification is carried out on a commercially available bushing. Firstly, the material parameters involved in Eq (8) are achieved from two measurements. Elastic and amplitude dependence parameters are obtained from a large deformation quasi-static test while the viscoelastic parameters are determined by a harmonic test performed over a frequency range from 1 to 100 Hz at an amplitude of 0.1 mm. Secondly, five new dynamic tests at 0.013, 0.04, 0.17, 0.25 and 0.35 mm are conducted in order to demonstrate that the formula for the axial stiffness predicts well the bushing behavior including the amplitude and frequency dependence. The test object is a commercial bushing manufactured by Trelleborg AVS under the trade name of VP 40/70120 with a rubber hardness of 60° IRH. The rubber type is a mixture between SMR 10 and SMR GP and the carbon black filler is a combination of N660 and N774. Instruments used for the measurements are displayed in Table 1.

**Setup**

The test component is mounted in a servo-hydraulic test machine consisting of a lower body with a hydraulic piston inside, two columns to the sides and an upper rigid crosshead, as shown in Figures 3 and 4. Measurements are performed with the inner surface of the bushing fixed and joined to the upper crosshead through a shaft and one load sensor, as displayed in Figure 3, while the outer surface of the bushing is subjected to axial displacements produced by the lower hydraulic piston. The motion of the bushing is measured by two displacement sensors symmetrically positioned on the outer surface of the bushing. The electrical signals are conditioned in a 6-channels amplifier producing electrical inputs to the frequency analyzer which are processed subsequently in a personal computer. In order to achieve constant amplitude, the electric signal that comes from the displacement sensor of the test machine and goes through the amplifier is automatically adjusted in a control loop in the analyzer to supply the exciter.

**Table 1 Measurement instruments**

<table>
<thead>
<tr>
<th>Instruments</th>
<th>Model</th>
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<tbody>
<tr>
<td>Servo-hydraulic piston</td>
<td>Instron 8032</td>
</tr>
<tr>
<td>6 channels Amplifier</td>
<td>HBM ML55B AB22A</td>
</tr>
<tr>
<td>LVDT displacement sensor</td>
<td>Schaeftz O50 MHR</td>
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<tr>
<td>Linear position sensor</td>
<td>HBM WETA 1/2 mm/E22267</td>
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<tr>
<td>Force transducer</td>
<td>AA-5 (0–5 kN) 4 POL/218 k</td>
</tr>
<tr>
<td>Frequency analyzer</td>
<td>Siglab 20–42</td>
</tr>
<tr>
<td>Computer</td>
<td>Aquila Pentium III MMX</td>
</tr>
</tbody>
</table>

**Figure 3 Test rig and measurement set-up diagram**
Measurements

The test component is mechanically conditioned before measurements to eliminate influence of the Mullins’ effect [20] which is shown when previously unstrained rubber is subjected to strain cycles at constant peak value reducing peak stress values at the first few oscillations.

Quasi-static test. Measurements at 0.1 Hz and 1 mm displacement amplitude are used to fit the elastic and amplitude dependence parameters. The large deformation assures that all friction has been developed in the loop and the low frequency guarantees that all viscous effects are eliminated.

Dynamic test. The bushing is excited by stepped sine displacements starting at 1 Hz and increasing with a constant frequency step of 1 Hz to a maximum frequency of 100 Hz, with amplitude held constant at 0.1 mm. Viscelastic parameters are the results of an iterative minimization process defined in Eq (9) between the complex stiffness obtained with one dynamic test performed from 1 to 100 Hz at 0.1 mm constant amplitude and the stiffness calculated by Eq (8) at the same conditions. In particular, when the viscoelastic parameters are assigned the values: $m = 6.0 \times 10^5$ Ns/m² and $\alpha = 0.20$, the measured stiffness in magnitude and loss factor is almost exactly reproduced by the model, as shown in Figure 6. The slight loss factor discrepancies at very low frequencies(~ 1 Hz) may be attributed to difficulties to measure low frequency phase angles. In addition, five new dynamic tests are performed at 0.013, 0.04, 0.17, 0.25 and 0.35 mm and the results are compared to those of the model when using the five material parameters calculated previously, displaying a good agreement over the whole frequency range, as presented in Figs. 7 and 8, which verifies the accuracy of Eq (8) representing the dynamic stiffness of a rubber bushing including the amplitude and frequency dependence.

Results and discussions

Calculations and graphical representations are carried out in MATLAB. Elastic and amplitude dependence parameters are varied until the model hysteresis loop enclosed by the resultant force in the bushing when is subjected to axial displacement at 0.1 Hz and 1 mm amplitude and such displacement fits well with the hysteresis loop obtained from one quasi-static test performed on the bushing at the same displacement conditions. The good agreement between the two loops displayed in Figure 5, is achieved with the values: $\mu = 2.4 \times 10^4$ N/m², $\sigma_{\text{max}} = 19200$ N/m² and $\varepsilon_{1/2} = 0.004$. Moreover, viscoelastic parameters are obtained from the error minimization iteration process defined in Eq (9) between the complex stiffness obtained with one dynamic test performed from 1 to 100 Hz at 0.1 mm constant amplitude and the stiffness calculated by Eq (8) at the same conditions. In particular, when the viscoelastic parameters are assigned the values: $m = 6.0 \times 10^5$ Ns/m² and $\alpha = 0.20$, the measured stiffness in magnitude and loss factor is almost exactly reproduced by the model, as shown in Figure 6. The slight loss factor discrepancies at very low frequencies (~ 1 Hz) may be attributed to difficulties to measure low frequency phase angles. In addition, five new dynamic tests are performed at 0.013, 0.04, 0.17, 0.25 and 0.35 mm and the results are compared to those of the model when using the five material parameters calculated previously, displaying a good agreement over the whole frequency range, as presented in Figs. 7 and 8, which verifies the accuracy of Eq (8) representing the dynamic stiffness of a rubber bushing including the amplitude and frequency dependence.

Conclusions

A simple and effective model for the axial dynamic stiffness of filled rubber bushings in the frequency domain including amplitude and frequency dependence is presented. It is developed by applying a material model to an equivalent strain of the strain state inside the bushing, thus yielding an equivalent modulus which is inserted into an analytical formula for the axial stiffness obtained by the classical theory of elasticity. An experimental verification is carried out on a commercially available bushing. Firstly, the model parameters are obtained from two tests: a quasi-static and dynamic, and subsequently five new dynamic experiments are performed on the bushing to
compare the results with those obtained with the developed formula at the same amplitude and frequency conditions, showing a good agreement which verifies the accuracy of the model.

Unlike previous simplified models, this method takes into account the amplitude dependence inside the rubber bushing due to non-homogeneous strain states. In addition, unlike tedious finite element models, this model as a function of the geometry and only five material parameters is a fast engineering tool to determine the most suitable rubber bushing to fulfil user requirements. This simplicity is necessary when a rubber bushing is only a small component in more complex systems where the dynamic behaviour is to be analysed.

A possible extension of this work is to carry out a similar process for the radial deformation in order to create a model representing the dynamic stiffness of the bushing in all directions which might be easily inserted into multi-body or finite elements systems.

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